Volatility Index: VIX vs. GVIX

"Does VIX Truly Measure Return Volatility?"
by Victor Chow, Wanjun Jiang, and Jingrui Li (2014)

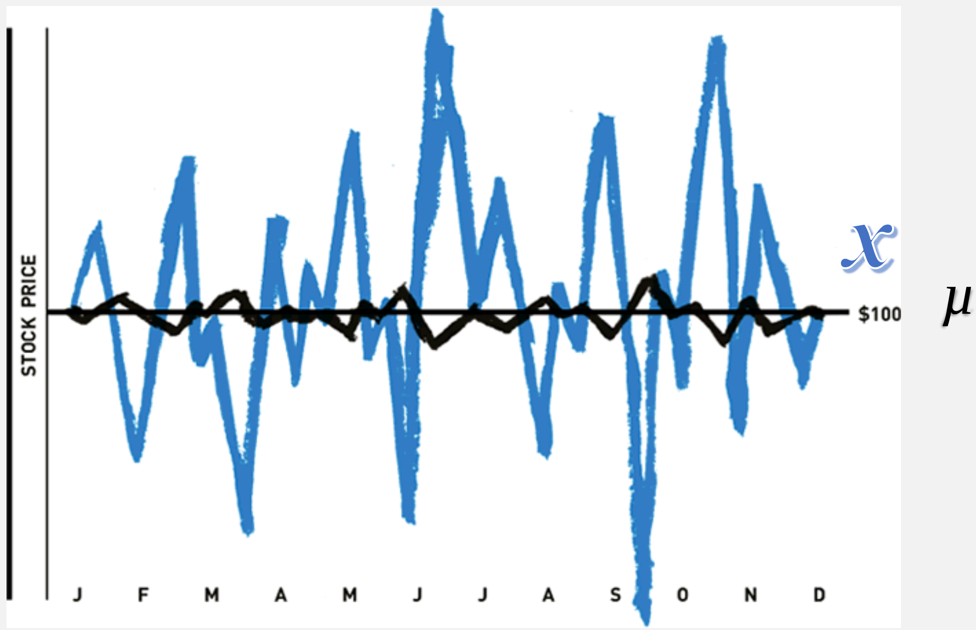
I.	**An Ex-ante (*forward-looking*) approach based on Market Price of Options; NOT an Ex-post (*backward-looking*) Statistical Estimation.**
II.	**NOT just Indexes, BUT Tradable Financial Instruments (See CBOE sites).**
III.	**Negatively Correlated with Underlying Index's Returns and thus Provide a good Market Risk Hedging Vehicle for Portfolio Management.**
IV.	**The Difference between GVIX and VIX indexes (GV-Spread) provides a good Forward-Looking Indicator about "Market Sentiment"**

Outlines

1. The Definition of Volatility
2. The Assumption of Symmetric Return Distribution
3. Geometric Brownian Motion: Foundation of the VIX
4. Core Derivation of VIX
5. Holding-Period Return, Log-Return, and Option Prices
6. Formulation of VIX
7. VIX is NOT a Volatility Index in general
8. GVIX is the True Volatility of Log-Returns
9. The GV-Spread (Empirical Evidence)
10. Correlation Matrix of GV-Spread and Distribution Moments
11. GV-Spread is Mean Reverting

Definition of Volatility

Volatility(σ)



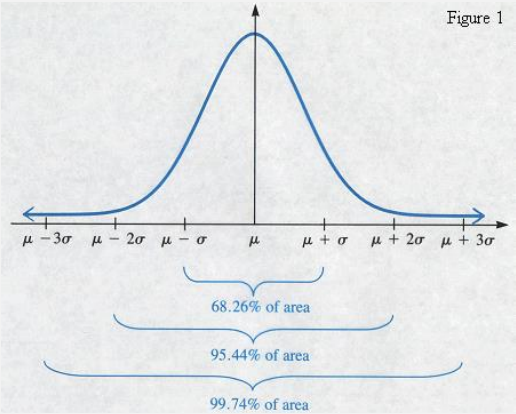
$$\mu = E(x); \qquad \sigma = \sqrt{E(x-\mu)^2} = \sqrt{E(x^2) - [\mu]^2}$$

CBOE VIX Formulation	**IS NOT BASED ON THE VOLATILITY DEFINITION.**
	BUT BASED ON THE ASSUMPTION OF SYMMETRIC RETURN DISTRIBUTION

The Assumption of Symmetric Return Distribution

Two-Moment Distribution

BELL CURVE

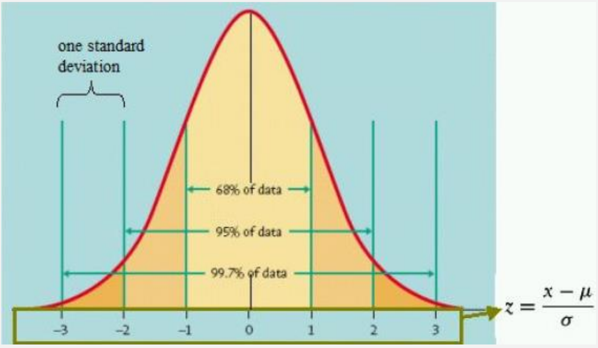


$$\tilde{x} = \mu_x + \sigma_x \tilde{z}$$

$$\tilde{y} = \mu_y + \sigma_y \tilde{z}$$

$$\tilde{x} \sim N(\mu_x, \sigma_x);\ \tilde{y} \sim N(\mu_y, \sigma_y)$$

STANDARIZED BELL CURVE



$$\tilde{z} = \frac{\tilde{x} - \mu_x}{\sigma_x}$$

$$\tilde{z} = \frac{\tilde{y} - \mu_y}{\sigma_y}$$

$$\tilde{z} \sim N(0,1)$$

Geometric Brownian Motion: Foundation of the VIX

Diffusion Process

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

1. $Z_0 = 0$
2. Z_t is almost surely everywhere continuous
3. Z_t has independent increments with $(Z_t - Z_s) \sim N(0, t-s)$ (for $0 \le s < t$)

$$\begin{aligned} d[\ln(S_t)] &= \boldsymbol{f'(S_t)dS_t + \frac{1}{2}f''(S_t)S_t^2\sigma^2 dt} \\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dZ_t) - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t \end{aligned}$$

Taylor Expansion (**Stop@2nd Order**) & Ito Calculus

Log returns follow a symmetric distribution

$$d[\ln(S_t)] = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t$$

A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the *logarithm* of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift.

put-call symmetry	"Classic put-call symmetry (Bowie and Carr 1994; Bates 1997) relates the prices of puts and calls at strikes that are unequal but equidistant logarithmically to the forward price. For example, it implies that if a forward price M follows ***geometric Brownian motion*** under an appropriate pricing measure, and $M_0 = 100$, then a 200-strike call on M has time-0 price equal to two times the price of the 50-strike put at the same expiry. " (see Peter Carr and Roger Lee, Put-Call Symmetry: Extension and Applications, *Mathematical Finance*, Vol. 19, No. 4 (October 2009), 523–560)

Core Derivation of VIX

1. Given that

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \text{and} \\ d[\ln(S_t)] = \left(\mu - \dfrac{1}{2}\sigma^2\right) dt + \sigma dZ_t \end{cases}$$

2. Solve for the volatility

$$\sigma^2\, dt = 2\left\{\frac{dS_t}{S_t} - d[\ln(S_t)]\right\}$$

3. Sum to T-period of time

$$\sigma^2 = \frac{1}{T}\int_0^T \sigma^2\, dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\left(\frac{S_T}{S_0}\right)\right]$$

4. Volatility Index (VIX)

$$\text{VIX}^2 = E(\sigma^2) = \sigma^2 = \frac{2}{T}\left[E\left(\int_0^T \frac{dS_t}{S_t}\right) - E\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right]$$

VIX in fact captures the difference between the **expected holding-period return** and the **expected log-return** over a T-period of time (e.g. 30-day)

Holding-Period Return, Log-Return, and Option Prices

1. Holding-Period Return $$R_T = \frac{S_T - S_0}{S_0}$$

2. Log-Return $$r_T = [\ln(S_T) - \ln(S_0)] = \ln\left(\frac{S_T}{S_0}\right)$$

3. Taylor Expansion with remainder. $$\ln(S_T) = \ln(S_0) + \frac{S_T - S_0}{S_0} + \int_{S_0}^{\infty} \frac{-1}{K^2}(S_T - K)^+ dK + \int_0^{S_0} \frac{-1}{K^2}(K - S_T)^+ dK$$

4. The difference between the two returns $$R_T - r_T = \left[\int_{S_0}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK + \int_0^{S_0} \frac{1}{K^2}(K - S_T)^+ dK\right]$$

5. The expected difference $$E(R_T) - E(r_T) = e^{rT}\left\{\int_{S_0}^{\infty} \frac{1}{K^2} C_T(K) dK + \int_0^{S_0} \frac{1}{K^2} P_T(K) dK\right\}$$

6. ***Expected Log-Return*** $$\boldsymbol{E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = E(R_T) + e^{rT}\left\{\int_{S_0}^{\infty} \frac{1}{K^2} C_T(K) dK + \int_0^{S_0} \frac{1}{K^2} P_T(K) dK\right\}}$$

Formulation of VIX

Recall

$$\mathrm{VIX}^2 = E(\sigma^2) = \sigma^2 = \frac{2}{T}\left[\int_0^T E\left(\frac{dS_t}{S_t}\right) - E\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right]$$

$$\left[\ln\left(\frac{S_T}{S_0}\right)\right] = E(R_T) + e^{rT}\left\{\int_{S_0}^{\infty}\frac{1}{K^2}C_T(K)dK + \int_0^{S_0}\frac{1}{K^2}P_T(K)dK\right\}$$

Under No-arbitrage

$$E\left(\int_0^T \frac{dS_t}{S_t}\right) = rT, \text{ and } E(R_T) = e^{rT} - 1$$

Volatility Index

$$\mathrm{VIX}^2 = \frac{2}{T}\left\{rT - (e^{rT} - 1) + e^{rT}\left[\int_{S_0}^{\infty}\frac{\mathbf{1}}{\boldsymbol{K^2}}\boldsymbol{C_T(K)dK} + \int_{\mathbf{0}}^{\boldsymbol{S_0}}\frac{\mathbf{1}}{\boldsymbol{K^2}}\boldsymbol{P_T(K)dK}\right]\right\}$$

$$= \frac{2}{T}\left\{rT - \left(\frac{F_0}{K_0} - 1\right) - \ln\left(\frac{K_0}{S_0}\right) + e^{rT}\left[\int_{K_0}^{\infty}\frac{1}{K^2}C_T(K)dK + \int_0^{K_0}\frac{1}{K^2}P_T(K)dK\right]\right\}$$

Taylor Expansion (Stop@2nd Order)

$$\left[rT - \left(\frac{F_0}{K_0} - 1\right) - \ln\left(\frac{K_0}{S_0}\right)\right] = \left[\ln\left(\frac{F_0}{K_0}\right) - \left(\frac{F_0}{K_0} - 1\right)\right] \approx -\frac{1}{2}\left(\frac{F_0}{K_0} - 1\right)^2$$

CBOE VIX Formulation

$$\mathrm{VIX}^2 = \frac{2e^{rT}}{T}\left[\int_{K_0}^{\infty}\frac{1}{K^2}C_T(K)dK + \int_0^{K_0}\frac{1}{K^2}P_T(K)dK\right] - \frac{1}{T}\left(\frac{F_0}{K_0} - 1\right)^2$$

$$\widehat{\mathrm{VIX}}^2 = \frac{2e^{rT}}{T}\sum_i \frac{1}{{K_i}^2}Q(K_i)\Delta K_i - \frac{1}{T}\left(\frac{F_0}{K_0} - 1\right)^2$$

VIX is NOT a Volatility Index in general

Key Component of VIX is $e^{rT}\left[\int_{S_0}^{\infty}\frac{1}{K^2}C_T(K)dK+\int_{0}^{S_0}\frac{1}{K^2}P_T(K)dK\right]$

It is the expected return difference.

$$E(R_T)-E(r_T)=e^{rT}\left\{\int_{S_0}^{\infty}\frac{1}{K^2}C_T(K)dK+\int_{0}^{S_0}\frac{1}{K^2}P_T(K)dK\right\}$$

Taylor Expansion (Stop@ N^{th} Order)

$$(1+R_T)=\frac{S_T}{S_0}=exp\left[\ln\left(\frac{S_T}{S_0}\right)\right]=1+\sum_{\kappa=1}^{N}\frac{1}{\kappa!}\left[\ln\left(\frac{S_T}{S_0}\right)\right]^{\kappa}+o\left[\ln\left(\frac{S_T}{S_0}\right)\right]^{N}$$

The expected return difference is

$$E(R_T)-E(r_T)=\frac{1}{2}E(r_T^2)+\frac{1}{6}E(r_T^3)+\frac{1}{24}E(r_T^4)+o[E(r_T^4)]$$

Key Component of VIX is $\frac{1}{2}E(r_T^2)+\frac{1}{6}E(r_T^3)+\frac{1}{24}E(r_T^4)+o[E(r_T^4)]$

Let $V_T=E(r_T^2),\quad W_T=E(r_T^3), and \quad X_T=E(r_T^4)$

VIX is a Moment-Combination

$$\text{VIX}=\frac{1}{\sqrt{T}}\sqrt{\left[V_T+\frac{W_T}{3}+\frac{X_T}{12}+o(X_T)\right]-2[(e^{rT}-1)-rT]}$$

GVIX is the True Volatility of Log-Returns

Definition of Variance

$$\mu = E(x), \qquad \boldsymbol{\sigma^2} = E(x-\mu)^2$$
$$V = E(x^2) \qquad = E(x^2) - [E(x)]^2 = \boldsymbol{V - \mu^2}$$

The Generalized Volatility Index (GVIX)

$$\mathbf{GVIX} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{T}}}\sqrt{\boldsymbol{V_T - (\mu_T)^2}}$$

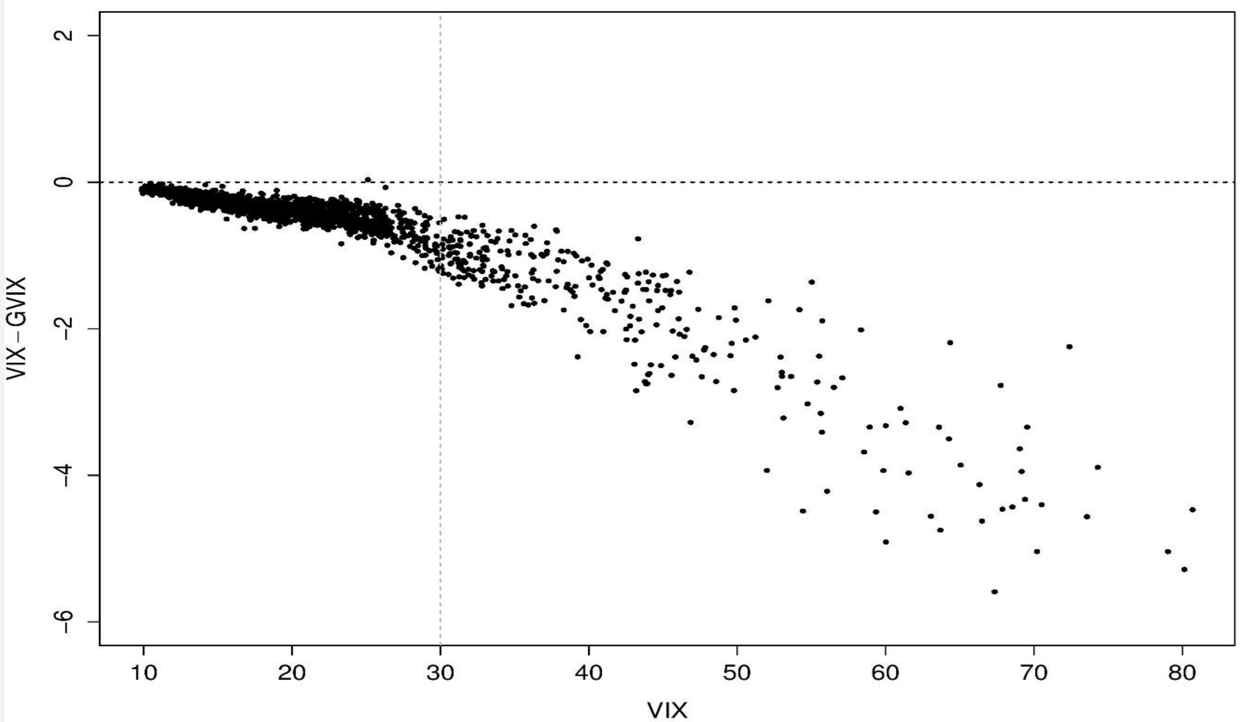
$$\mu_T = \ln\left(\frac{K_0}{S_0}\right) + \left(\frac{F_0}{K_0} - 1\right) - e^{rT}\left[\int_{K_0}^{\infty}\frac{1}{K^2}C_T(K)dK + \int_0^{K_0}\frac{1}{K^2}P_T(K)dK\right]$$

$$V_T = \ln^2\left(\frac{K_0}{S_0}\right) + 2\ln\left(\frac{K_0}{S_0}\right)\left(\frac{F_0}{K_0} - 1\right)$$
$$+ 2e^{rT}\left[\int_{K_0}^{\infty}\frac{\left[1 - \ln\left(\frac{K}{S_0}\right)\right]}{K^2}C_T(K)dK + \int_0^{K_0}\frac{\left[1 + \ln\left(\frac{S_0}{K}\right)\right]}{K^2}P_T(K)dK\right]$$

GV-Spread (Empirical Evidence)

The Spread	$\mathrm{VIX}^2 - \mathrm{GVIX}^2 \approx \frac{1}{T}\left(\mu_T^2 + \frac{W_T}{3} + \frac{X_T}{12}\right)$

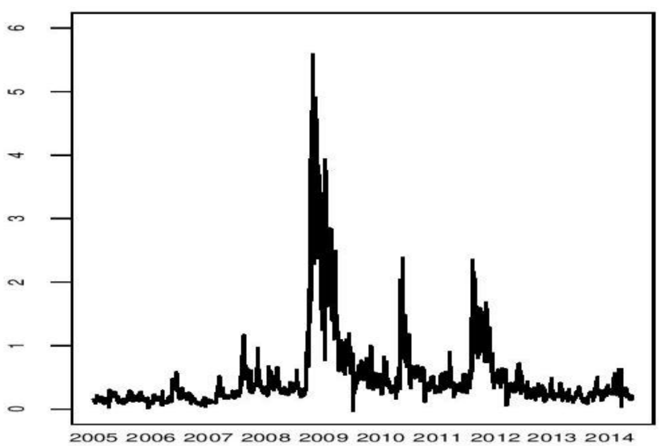
GVIX > VIX indicates that $\boldsymbol{W_T < 0}$



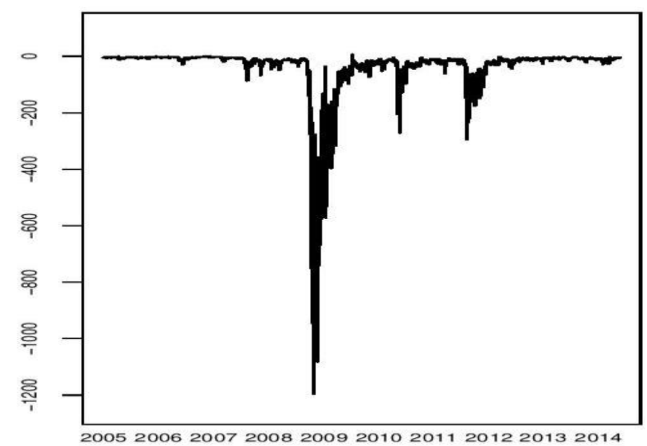
$$\boldsymbol{W_T} = \ln^3\left(\frac{K_0}{S_0}\right) + 3\ln^2\left(\frac{K_0}{S_0}\right)\left(\frac{F_0}{K_0} - 1\right) + 3e^{rT}\left[\int_{K_0}^{\infty} \frac{\left[2\ln\left(\frac{K}{S_0}\right) - \ln^2\left(\frac{K}{S_0}\right)\right]}{K^2}\boldsymbol{C_T(K)}dK - \int_0^{K_0} \frac{\left[2\ln\left(\frac{S_0}{K}\right) + \ln^2\left(\frac{S_0}{K}\right)\right]}{K^2}\boldsymbol{P_T(K)}dK\right]$$

$$\mathrm{VIX}^2 - \mathrm{GVIX}^2 \approx \frac{1}{T}\left(\mu_T^2 + \frac{W_T}{3} + \frac{X_T}{12}\right)$$

A. $\widehat{\mathrm{GVIX}}_a - \widehat{\mathrm{VIX}}_a$ (Spread)



B. $\frac{1}{3}\widehat{\mathrm{W}}_a$



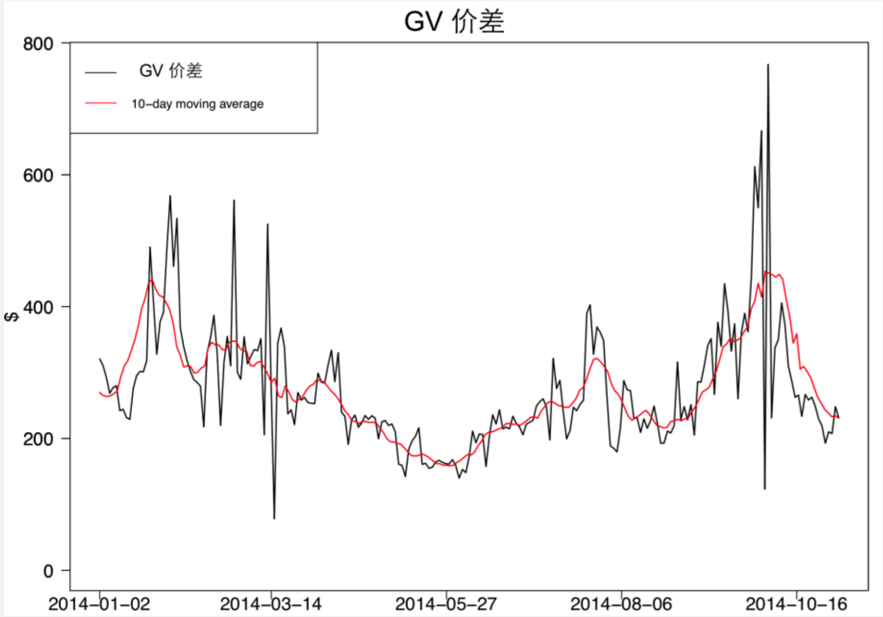
Correlation Matrix of GV-Spread and Distribution Moments

	$\widehat{V}_a$	$\widehat{W}_a$	$\widehat{X}_a$
$\widehat{GVIX}_a - \widehat{VIX}_a$	0.96	-0.97	0.90
$\widehat{V}_a$		-0.95	0.86
$\widehat{W}_a$			-0.97

$\widehat{V}_a$	daily (annualized) ex-ante second moment (Volatility)
$\widehat{W}_a$	daily (annualized) ex-ante third moment (Skewness)
$\widehat{X}_a$	daily (annualized) ex-ante fourth moment (Kurtosis)

GV-Spread is Mean-Reverting

The $Spread Chart (2014)



Zivot-Andrews Unit-Root Test