$$\begin{cases} \tilde{R}_j = \alpha_j + \beta_j \tilde{R}_m + \tilde{\varepsilon}_j & \text{(market model)} \\ \bar{\varepsilon}_j = 0, \quad Cov(\tilde{R}_m, \tilde{\varepsilon}_j) = \sigma_{m,\varepsilon_j} = \phi = \overline{R_m \varepsilon_j} - \underbrace{\cancel{\bar{R}_m \cdot \bar{\varepsilon}_j}}_{=0} \end{cases}$$

$$\bar{R}_j = \alpha_j + \beta_j \bar{R}_m + 0 \; ; \quad ① \; \alpha_j = \bar{R}_j - \beta_j \bar{R}_m$$

$$R_j R_m = \alpha_j R_m + \beta_j R_m^2 + R_m \varepsilon_j$$

Ⓐ $\overline{R_j R_m} = \alpha_j \bar{R}_m + \beta_j \overline{R_m^2} + \underbrace{\cancel{\overline{R_m \varepsilon_j}}}_{=0}$

Ⓑ $\bar{R}_j \cdot \bar{R}_m = \alpha_j \bar{R}_m + \beta_j \bar{R}_m \cdot \bar{R}_m$

Ⓐ − Ⓑ, we have

$$\overline{R_j R_m} - \bar{R}_j \cdot \bar{R}_m = \beta_j \left[\overline{R_m^2} - \bar{R}_m \cdot \bar{R}_m \right]$$

$$Cov(R_j, R_m) = \beta_j \sigma_m^2$$

Thus,

② $$\beta_j = \frac{Cov(R_j \cdot R_m)}{\sigma_m^2} = \frac{\sigma_j \sigma_m \cdot \rho_{j,m}}{\sigma_m^2} = \frac{\sigma_j}{\sigma_m} \rho_{j,m}$$

Single Index Model

$$\sigma_j^2 = \overline{R_j^2} - \bar{R}_j^2 \; ; \qquad (1)\; \overline{R_j^2} = \overline{(\alpha_j + \beta_j R_m + \varepsilon_j)^2}$$

$$= \alpha_j^2 + \beta_j^2 \overline{R_m^2} + \overline{\varepsilon_j^2} + 2\alpha_j \beta_j \overline{R_m} + \cancel{2\alpha_j \overline{\varepsilon_j}}^{=0} + \cancel{2\beta_j \overline{R_m \varepsilon_j}}_{=0}$$

$$(2)\; \bar{R}_j^2 = (\alpha_j + \beta_j \bar{R}_m)^2$$

$$= \alpha_j^2 + \beta_j^2 \bar{R}_m^2 + 2\alpha_j \beta_j \bar{R}_m$$

$$\sigma_j^2 = (1) - (2) = \beta_j^2 \left[\overline{R_m^2} - \bar{R}_m^2\right] + \left[\overline{\varepsilon_j^2} - \underbrace{\bar{\varepsilon}_j^2}_{=0}\right]$$

Risk Decomposition:

$$\sigma_j^2 = \left[\beta_j^2 \sigma_m^2\right] + \left[\sigma_{\varepsilon_j}^2\right]$$

market Variance

Residual Variance

Total Risk = [Systematic Risk] + [Unsystematic Risk]

Diversifiable

idiosyncratic Risk

residual risk

Single Index Model

$$R^2 = \text{Coefficient of Determination} = \rho_{jm}^2 = \frac{\text{Systematic Risk}}{\text{Total Risk}}$$

R-square

$$= \frac{\beta_j^2 \cdot \sigma_m^2}{\sigma_j^2} = \left[\frac{\left(\frac{\cancel{\sigma_j}}{\cancel{\sigma_m}}\rho_{jm}\right)\cdot\cancel{\sigma_m}}{\cancel{\sigma_j}}\right]^2 = \rho_{jm}^2$$

if the correlation between asset j and the market pfl is 90%, then $\frac{\text{Systematic Risk of } j}{\text{Total Risk of } j} = 81\%$.

Thus, 19% of asset j's risk is Unsystematic.

R-square (R^2) is a statistical term of "goodness of fit".

Single Index Model

Portfolio Application: Assumption $\longrightarrow$ $Cov(\varepsilon_i, \varepsilon_j) = 0$ "Uncorrelated Residuals"

$$\boxed{\sigma_{ij}} = Cov(R_i, R_j) = E(R_i R_j) - E(R_i)E(R_j) = \overline{R_i R_j} - \overline{R_i}\,\overline{R_j}$$

$$= \overline{(\alpha_i + \beta_i R_m + \varepsilon_i)(\alpha_j + \beta_j R_m + \varepsilon_j)} - \overline{(\alpha_i + \beta_i R_m + \varepsilon_i)} \cdot \overline{(\alpha_j + \beta_j R_m + \varepsilon_j)}$$

Note:

$Cov(R_m, \varepsilon_i) = 0$

$Cov(R_m, \varepsilon_j) = 0$

$$= \left[\cancel{\alpha_i\alpha_j} + \cancel{\alpha_i\beta_j\overline{R_m}} + \underbrace{\cancel{\alpha_i\overline{\varepsilon_j}}}_{=0} + \cancel{\alpha_j\beta_i\overline{R_m}} + \beta_i\beta_j\overline{R_m^2} + \underbrace{\cancel{\beta_i\overline{R_m\varepsilon_j}}}_{=0} + \underbrace{\cancel{\overline{\varepsilon_i}\alpha_j}}_{=0} + \underbrace{\cancel{\beta_j\overline{R_m\varepsilon_i}}}_{=0} + \underbrace{\cancel{\overline{\varepsilon_i\varepsilon_j}}}_{=0} \right] - \left[\cancel{\alpha_i\alpha_j} + \cancel{\alpha_i\beta_j\overline{R_m}} + \cancel{\alpha_i\overline{\varepsilon_j}} + \cancel{\alpha_j\beta_i\overline{R_m}} + \beta_i\beta_j\overline{R_m}^2 + \underbrace{\cancel{\beta_i\overline{R_m}\cdot\overline{\varepsilon_j}}}_{=0} + \cancel{\overline{\varepsilon_i}\alpha_j} + \underbrace{\cancel{\beta_j\overline{R_m}\cdot\overline{\varepsilon_i}}}_{=0} + \underbrace{\cancel{\overline{\varepsilon_i}\cdot\overline{\varepsilon_j}}}_{=0} \right]$$

$$= \beta_i\beta_j\overline{R_m^2} - \beta_i\beta_j\overline{R_m}^2$$

$$= \boxed{\beta_i\beta_j\sigma_m^2}$$

$$= \frac{\sigma_i}{\cancel{\sigma_m}}\rho_{im} \cdot \frac{\sigma_j}{\cancel{\sigma_m}}\rho_{jm} \cdot \cancel{\sigma_m^2} = \boxed{\sigma_i\sigma_j\rho_{im}\rho_{jm}}$$

Single Index Model

$R_p = \sum \omega_j R_j$; $R_j = \alpha_j + \beta_j R_m + \varepsilon_j$; where $\bar{\varepsilon}_j = \mathrm{Cov}(\varepsilon_j, R_m) = \mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0$

$$\beta_p = \frac{\mathrm{Cov}(R_p, R_m)}{\sigma_m^2} = \frac{\overline{R_p R_m} - \bar{R}_p \cdot \bar{R}_m}{\sigma_m^2}$$

$$= \frac{\sum \omega_j \overline{R_j R_m} - \sum \omega_j \bar{R}_j \cdot \bar{R}_m}{\sigma_m^2}$$

$$= \sum \omega_j \left(\frac{\overline{R_j R_m} - \bar{R}_j \bar{R}_m}{\sigma_m^2} \right)$$

$$= \sum \omega_j \frac{\mathrm{Cov}(R_j, R_m)}{\sigma_m^2}$$

$$= \sum \omega_j \beta_j$$

$$R_p = \sum \omega_j R_j$$

$$= \sum \omega_j (\alpha_j + \beta_j R_m + \varepsilon_j)$$

$$= \underbrace{\sum \omega_j \alpha_j} + \underbrace{\sum \omega_j \beta_j} R_m + \underbrace{\sum \omega_j \varepsilon_j}$$

$$= \alpha_p + \beta_p R_m + \varepsilon_p$$

Thus,

$$\begin{cases} \bar{R}_p = \alpha_p + \beta_p \bar{R}_m + \cancel{\bar{\varepsilon}_p}^{=0} \\ \sigma_p^2 = \beta_p^2 \sigma_m^2 + \sigma_{\varepsilon p}^2 \end{cases}$$

Coefficient of Determination $= \rho_{p,m}^2$

Single Index Model

Why Beta coefficient (β_j) can be used for systematic risk measurement?

$$\text{Systematic Risk}_j = \beta_j^2 \sigma_m^2$$

$$\beta_j^2 \sigma_m^2 > \beta_i^2 \sigma_m^2 \iff \beta_j > \beta_i$$

$$\left(\frac{\sigma_j}{\cancel{\sigma_m}} \cdot \rho_{jm} \cdot \cancel{\sigma_m}\right)^2 > \left(\frac{\sigma_i}{\cancel{\sigma_m}} \rho_{im} \cancel{\sigma_m}\right)^2 \iff \sigma_j \rho_{jm} > \sigma_i \rho_{im}$$