	Expected Annual Return ($\bar{R}$)	Standard Deviation of Returns (σ)
Asset 1	19%	33%
Asset 2	8%	13%

How to form an optimal portfolio; if there is $100,000 capital available investment?

→ Depends on the Target Risk and Return!
Depends on the Correlation between Assets!

Case 1: ① Target Risk of portfolio $= 0 = \sigma_p$

② Correlation between Asset 1 & Asset 2 $= \rho_{12} = -1$

" Negatively and Perfectly Correlated "

Correlation and Portfolio Formation

Consider

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

Note: $\sigma_{12} = \sigma_1\sigma_2\rho_{12}$

$$= (w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2(w_1\sigma_1)(w_2\sigma_2)\rho_{12}$$

$\boxed{\text{Given that } \rho_{12} = -1}$

$$= (w_1\sigma_1)^2 + (w_2\sigma_2)^2 - 2(w_1\sigma_1)(w_2\sigma_2)$$

$$= (w_1\sigma_1 - w_2\sigma_2)^2$$

$\boxed{\text{Target}: \ \sigma_p^2 = 0}$; So $(w_1\sigma_1 - w_2\sigma_2) = 0$

$\boxed{w_1 + w_2 = 1 \text{, or } w_2 = 1 - w_1}$

Thus,

$$w_1\sigma_1 = (1-w_1)\sigma_2 \ ; \quad w_1(\sigma_1+\sigma_2) = \sigma_2$$

$$w_1 = \frac{\sigma_2}{\sigma_1+\sigma_2} \qquad w_2 = 1 - w_1 \doteq 72\%$$

$$= \frac{13\%}{33\% + 13\%} = \overset{28}{\cancel{80}}\%$$

Case 1: $\rho = -1$

Invest \$ 28,000 in Asset 1	Invest \$ 72,000 in Asset 2

Case 2 ① Target : $\sigma_p = 0$

② $\rho_{12} = +1$

$$\sigma_p^2 = (w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12}$$

$$\phi = (w_1\sigma_1)^2 + ((1-w_1)\sigma_2)^2 + 2(w_1\sigma_1)[(1-w_1)\sigma_2]$$

$$\phi = (w_1\sigma_1 + (1-w_1)\sigma_2)^2$$

Then, $w_1\sigma_1 + (1-w_1)\sigma_2 = 0 = w_1(\sigma_1 - \sigma_2) + \sigma_2$

Thus, $w_1(\sigma_1 - \sigma_2) = -\sigma_2$; $\begin{cases} w_1 = \dfrac{-\sigma_2}{\sigma_1 - \sigma_2} = \dfrac{-13\%}{33\% - 13\%} = -65\% \\ w_2 = 1 - w_1 = 165\% \end{cases}$

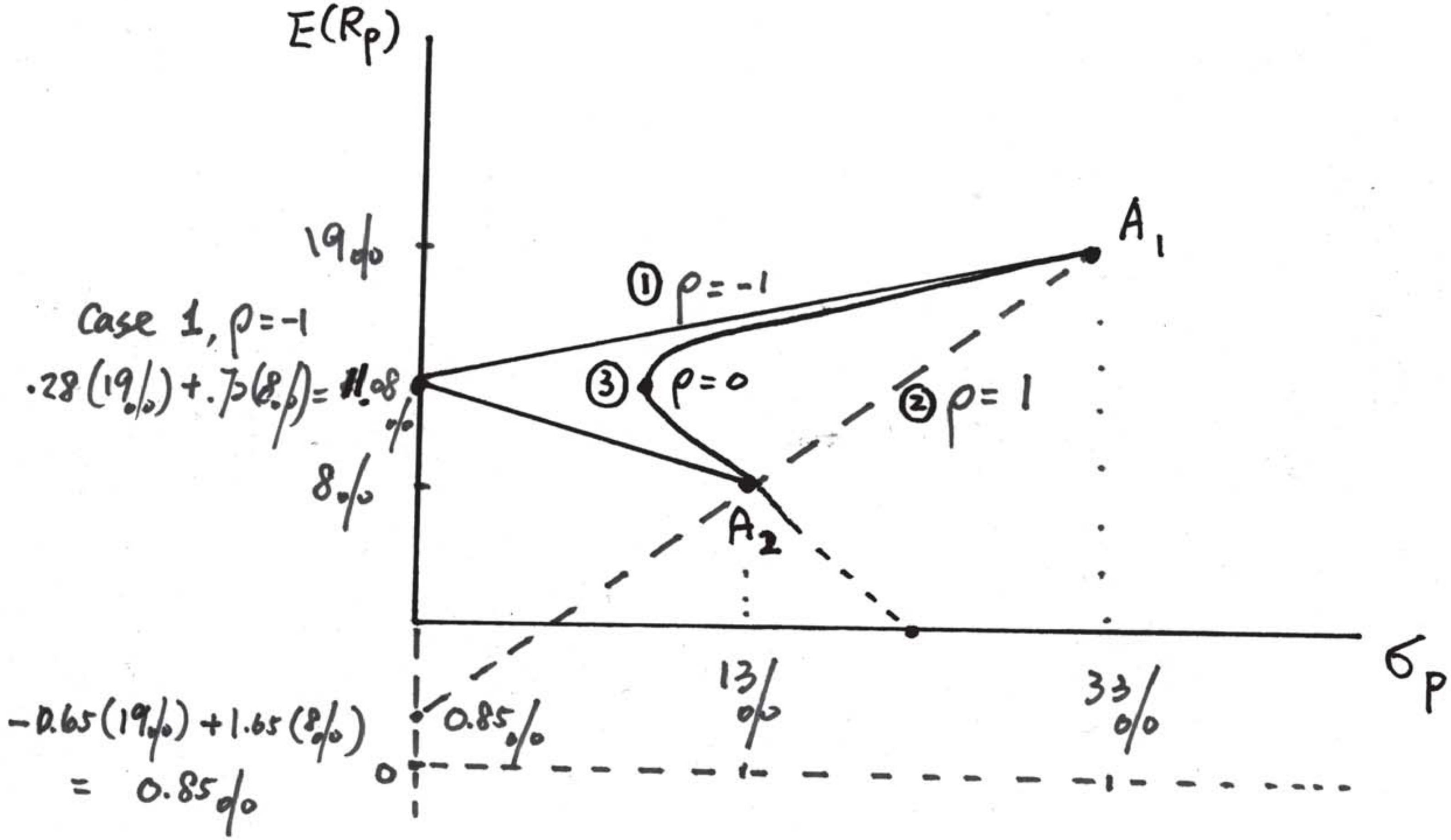
Net Investment (Equity) = \$100,000

Short Sell \$65,000 from Asset 1	Invest \$165,000 in Asset 2

Case 2

$\rho = +1$

Correlation and Portfolio Formation



* if $\rho_{12} = -1$, then No short sell is required to meet the target

if $\rho_{12} = 1$, then Short sell is required.

if $-1 < \rho < 1$, then $\sigma_p = 0$ is <u>NOT</u> an optimal target portfolio.

Correlation and Portfolio Formation

Optimal portfolio (minimum σ^2 pfl) of 2 Assets in general $(-1 < \rho_{12} < 1)$:

$$\sigma_p^2 = \underbrace{w_1^2\sigma_1^2 + w_2^2\sigma_2^2}_{w_2^2\sigma_2^2 \,=\, (1-w_1)^2\sigma_2^2} + \underbrace{2w_1 w_2 \sigma_1\sigma_2\rho_{12}}_{=\,2w_1(1-w_1)\sigma_1\sigma_2\rho_{12}}$$

$\Rightarrow$ minimize σ_p^2 $\Rightarrow$ $\boxed{\dfrac{d\sigma_p^2}{dw_1} = 0}$ F.O.C.

$$\frac{d\sigma_p^2}{dw_1} = 0 = 2w_1\sigma_1^2 + 2w_1\sigma_2^2 - 2\sigma_2^2 + 2\,\cancel{w_1}\,\sigma_1\sigma_2\rho_{12} - 4w_1\sigma_1\sigma_2\rho_{12}$$

So,
$$w_1 = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}} = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

$$w_2 = 1 - w_1$$

Special Case $(\rho_{12} = 0)$, Then,
$$\begin{cases} w_1 = \dfrac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \\ 1 - w_1 = w_2 \end{cases}$$

Correlation and Portfolio Formation

Case 3 : $\rho_{12} = 0$, Optimal portfolio ?

$$W_1 = \frac{(.33)^2}{(.13)^2 + (.33)^2} = \frac{.1089}{.0169 + .1089} = 86.6\%$$

$$W_2 = 13.4\%$$

$$E(R_p) = (.866)(19\%) + (.134)(8\%) = 17.53\%$$

Case 4 : $\rho_{12} = 0.85$

$$W_1 = \frac{(.33)^2 - (.33 \times .13 \times .85)}{(.13)^2 + (.33)^2 - 2(.13 \times .33 \times .85)} = \frac{.1089 - .0365}{.0169 + .1089 - .073} = \frac{.0724}{.0528} = 1.37$$

$$W_2 = 1 - 1.37 = -37\%$$

$$E(R_p) = (1.37)(19\%) - (.37)(8\%) = 23\%$$