Risk & Return : Part I "Portfolio Risk & Return Measure"

Portfolio = a combination (set) of assets based on different weights (or allocation of wealth or Capital)

The expected return of a portfolio ($E(R_p)$) is simply a weighted Sum value of individual assets' expected returns.

$$E(R_p) = \sum_{i=1}^{N} W_i E(R_i) \quad ; \quad \bar{R}_p = \sum_{i=1}^{N} W_i \bar{R}_i$$

However, $\sigma_p^2 \neq \sum_{i=1}^{N} w_i \, \sigma_{\cancel{p}i}^2$ ← The risk measure (σ^2) for a pfl is much more complicate.

Now, Let's consider that the varivance is the <u>Expected Value</u> of <u>deviation squar</u>

$$\sigma_x^2 = E\left(x - E(x)\right)^2$$

" Portfolio Risk & Return Measare "

For convenient purpose, Let $\overline{X} = E(x)$.

$$\sigma_x^2 = E(x - \overline{x})^2 = E(x^2) - [E(x)]^2$$

Suppose we have a portfolio that contants only two risky assets 1 & 2, respectively. Then

① $$\overline{R}_p = w_1 \overline{R}_1 + w_2 \overline{R}_2 = \sum_{i=1}^{2} w_i \cdot \overline{R}_i \quad ; \text{ where } w_1 + w_2 = 1$$

② $$\begin{aligned} \sigma_p^2 &= E(R_p - \overline{R}_p)^2 \\ &= E[(w_1 R_1 + w_2 R_2) - (w_1 \overline{R}_1 + w_2 \overline{R}_2)]^2 \\ &= E[w_1 (R_1 - \overline{R}_1) + w_2 (R_2 - \overline{R}_2)]^2 \end{aligned}$$

Note: $(x+y)^2 = x^2 + y^2 + 2xy$

"Portfolio Risk & Return Measure"

$$\sigma_p^2 = E\left[w_1(R_1-\bar{R}_1) + w_2(R_2-\bar{R}_2)\right]^2$$

$$= E\left[w_1^2(R_1-\bar{R}_1)^2 + w_2^2(R_2-\bar{R}_2)^2 + 2w_1w_2(R_1-\bar{R}_1)(R_2-\bar{R}_2)\right]$$

$$= w_1^2 \underbrace{E(R_1-\bar{R}_1)^2} + w_2^2 \underbrace{E(R_2-\bar{R}_2)^2} + 2w_1w_2 \underbrace{E\left[(R_1-\bar{R}_1)(R_2-\bar{R}_2)\right]}_{\text{Covariance}}$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

$$= \left[w_1w_1\sigma_{11}\right] + \left[w_2w_2\sigma_{22}\right] + \left[w_1w_2\sigma_{12} + w_2w_1\sigma_{21}\right]$$

Note: $\sigma_{12} = \sigma_{21}$

$$\begin{array}{c c} & \begin{array}{cc} w_1 \downarrow & w_2 \downarrow \end{array} \\ \begin{array}{c} w_1 \rightarrow \\ w_2 \rightarrow \end{array} & \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \end{array}$$

$\leftarrow$ Variance - Covariance

$$= \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \sigma_{ij}$$

$\leftarrow$ Weighted Sum of all pairs of variances and covariances among assets' returns.

" Portfolio Risk & Return Measure "

In general,

$$\sigma_p^2 = \begin{array}{c} \\ w_1 \rightarrow \\ w_2 \rightarrow \\ \\ w_N \rightarrow \end{array} \begin{array}{c} \begin{array}{cccc} \overset{w_1}{\downarrow} & \overset{w_2}{\downarrow} & & \overset{w_N}{\downarrow} \end{array} \\ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots\cdots & \sigma_{NN} \end{bmatrix} \end{array} ; \quad \text{where } \sum_{i=1}^{N} w_i = 1$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \, w_j \, \sigma_{ij}$$

$$= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i,j=1,\, i \neq j}^{N} w_i \, w_j \, \sigma_{ij}$$

(See Page 215 of the Text.)

Diversification

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i,j=1;\, i \neq j}^{N} w_i w_j \sigma_{ij}$$

Let $\bar{\sigma}^2$ = the average of σ_i^2 ; $\overline{Cov}$ = the average of σ_{ij}

$w_i = w_j = \frac{1}{N}$ ($\leftarrow$ equally weighted)

We have

$$\sigma_p^2 = \frac{\bar{\sigma}^2}{N} + \frac{(N-1)}{N} \overline{Cov}$$

When $N \to \infty$ (very large)

$$\sigma_p^2 \approx 0 + 1\,\overline{Cov} = \overline{Cov}$$ $\leftarrow$ Reducing Risk as $N \uparrow$

Covariance & Correlation

$$\sigma_{ij} = \sigma_i \cdot \sigma_j \, \rho_{ij} \iff \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

$$\sigma_p^2 = \sum_i^N \sum_j^N w_i w_j \sigma_{ij} = \sum_i^N \sum_j^N w_i w_j \sigma_i \sigma_j \rho_{ij}$$

if all assets have the same variance (σ^2) and the same Correlation (ρ) among assets in the equally weighted portfolio, then

$$\sigma_p^2 = \frac{\sigma^2}{N} + \frac{N-1}{N} \sigma^2 \rho \; ;$$

So, if N is large and Correlation of assets approch zero, then $\sigma_p^2 \sim \emptyset$